

# Connected Green function approach to ground state symmetry breaking in $\Phi_{1+1}^4$ -theory<sup>\*†</sup>

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Using the cluster expansions for n-point Green functions we derive a closed set of dynamical equations of motion for connected equal-time Green functions by neglecting all connected functions higher than  $4^{th}$  order for the  $\lambda\Phi^4$ -theory in  $1 + 1$  dimensions. We apply the equations to the investigation of spontaneous ground state symmetry breaking, i.e. to the evaluation of the effective potential at temperature  $T = 0$ . Within our momentum space discretization we obtain a second order phase transition (in agreement with the Simon-Griffith theorem) and a critical coupling of  $\lambda_{crit}/4m^2 = 2.446$  as compared to a first order phase transition and  $\lambda_{crit}/4m^2 = 2.568$  from the Gaussian effective potential approach.

## I. INTRODUCTION

In the last years there have been many attempts to develop non-perturbative methods for quantum field theoretical (QFT) problems such as the infrared behaviour of QCD. Many of the non-perturbative methods used so far, such as the coupled cluster expansion [1–3], originate from standard many-body theory and use states in the Fock representation, thus they have the general problem of Fock representations being disjunct to the Hilbert space of QFT [3,4].

In this work we also adopt a successful method from standard many-body theory, which however is formulated in terms of Green functions and therefore of a genuine field-theoretical nature. We propose an approach along the line of n-body correlation dynamics [5], which describes the propagation of the system in terms of equal-time Green functions (i.e. we use the density matrix formalism). The basic strategy of correlation dynamics is the truncation of the infinite hierarchy of equations of motion for n-point Green functions via the use of cluster expansions, i.e. the expansion of Green functions in terms of connected Green functions (correlation functions). In this work we go up to the connected 4-point level, i.e. we include the 2-,3- and 4-field correlation functions. For relativistic systems with a nonlocal interaction our method does not guarantee a covariant description, since retardation effects can not be included due to the equal-time formalism. There is, however, no objection against the use of an equal-time formalism in the case of a local relativistic field theory as long as one consistently considers Green functions containing canonically conjugate field momenta as well as the fields themselves.

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This paper is organized as follows: In section II we describe the derivation of the correlation dynamical equations of motion for  $\Phi^4$ -theory in  $1+1$  dimensions; the final set of equations itself is shifted to appendix A in view of its length. Section III is devoted to the application of the method to the evaluation of the effective potential at zero temperature within various limits and thus to the investigation of ground state symmetry breaking.

## II. CORRELATION DYNAMICS FOR $\Phi_{1+1}^4$ -THEORY

We consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m_0^2 \phi^2 - \frac{1}{4} \lambda \phi^4, \quad (\text{II.1})$$

in one space and one time dimension, which corresponds to the Hamiltonian

$$H = \frac{1}{2} \int dx \left[ \pi^2 + \left( \frac{\partial \phi}{\partial x} \right)^2 + m_0^2 \phi^2 + \frac{1}{2} \lambda \phi^4 \right], \quad (\text{II.2})$$

where  $\pi = \partial_t \phi$  and  $m_0$  is the bare mass of the scalar field  $\phi$ . In order to evaluate the effective potential of the theory at zero temperature, i.e. the minimum of the energy density for a given magnetization  $\langle \phi \rangle$ , we decompose  $\phi$  into a classical and a quantum part according to

$$\phi = \Phi_0 + \Phi, \quad (\text{II.3})$$

where  $\Phi_0$  is a real constant and assume

$$\langle \Phi \rangle = 0, \quad \text{i.e.} \quad \langle \phi \rangle = \Phi_0. \quad (\text{II.4})$$

Since  $\Phi_0$  is a constant, we have

$$\pi = \partial_t \phi = \partial_t \Phi = \Pi. \quad (\text{II.5})$$

The Hamiltonian, expressed in terms of the classical part  $\Phi_0$  and the quantum part  $\Phi$  of  $\phi$ , then reads

$$H = \frac{1}{2} \int dx \left[ \Pi^2 + \left( \frac{\partial \Phi}{\partial x} \right)^2 + (2\Phi_0 m_0^2 + 2\lambda \Phi_0^3) \Phi + (m_0^2 + 3\lambda \Phi_0^2) \Phi^2 + 2\lambda \Phi_0 \Phi^3 + \frac{1}{2} \lambda \Phi^4 + m_0^2 \Phi_0^2 + \frac{1}{2} \lambda \Phi_0^4 \right]. \quad (\text{II.6})$$

For the time evolution of  $\Phi$  and  $\Pi$  we obtain with (II.6) and the canonical equal-time commutation relations

$$[\Phi(x, t), \Phi(y, t)] = [\Pi(x, t), \Pi(y, t)] = 0, \quad [\Phi(x, t), \Pi(y, t)] = i\delta(x - y) \quad (\text{II.7})$$

by means of the Heisenberg equation:

$$\begin{aligned} \partial_t \Phi &= \Pi, \\ \partial_t \Pi &= -(\Phi_0 m_0^2 + \lambda \Phi_0^3) + (\partial_x^2 - m_0^2 - 3\lambda \Phi_0^2) \Phi - 3\lambda \Phi_0 \Phi^2 - \lambda \Phi^3. \end{aligned} \quad (\text{II.8})$$

In analogy to (II.8) we get the equations of motion for equal-time operator products of  $\Phi$  and  $\Pi$ , e.g.

$$\begin{aligned}\partial_t (\Phi(x_1, t)\Phi(x_2, t)) &= \Pi(x_1, t)\Phi(x_2, t) + \Phi(x_1, t)\Pi(x_2, t) , \\ \partial_t (\Pi(x_1, t)\Phi(x_2, t)) &= [- (\Phi_0 m_0^2 + \lambda \Phi_0^3) + (\partial_x^2 - m_0^2 - 3\lambda \Phi_0^2) \Phi(x_1, t) \\ &\quad - 3\lambda \Phi_0 \Phi^2(x_1, t) - \lambda \Phi^3(x_1, t)] \Phi(x_2, t) + \Pi(x_1, t)\Pi(x_2, t) , \quad \dots \quad .\end{aligned}\tag{II.9}$$

After taking their expectation values, the equations of motion for all possible n-point equal-time products of  $\Phi$  and  $\Pi$  comprise an infinite hierarchy of equations of motion for equal-time Green functions. The analogon of this in nonrelativistic many-body theory is the BBGKY<sup>1</sup> density matrix hierarchy [6]. Since we are considering a local field theory, the equations of motion for the Green functions contain no retardation integrals, which implies that working in an equal-time limit is sufficient for describing the propagation of the system as long as one considers all Green functions containing the fields as well as their conjugate momenta.

For practical purposes, the infinite hierarchy of coupled differential equations of first order in time has to be truncated. This is done using the cluster expansions for n-point Green functions [7], i.e. their decomposition into sums of products of connected Green functions, which works as long as the system is in a pure phase [8,9]. The explicit form of the cluster expansions can be derived from the generating functionals of full and connected Green functions,  $Z[J, \sigma]$  and  $W[J, \sigma]$ , given by

$$Z[J, \sigma] = \text{Tr} \left\{ \rho T \left[ e^{i \int d^2 \hat{x} (J(\hat{x}) \Phi(\hat{x}) + \sigma(\hat{x}) \Pi(\hat{x}))} \right] \right\} \quad \text{and} \quad Z[J, \sigma] = e^{W[J, \sigma]} ,\tag{II.10}$$

respectively, where  $T$  is the time ordering operator,  $\rho$  is the statistical density operator describing the pure or mixed state of the system ( $\text{Tr} \rho = 1$ ) and  $\hat{x} = (x, t)$ . We start with the cluster expansions for the time-ordered Green functions with different time arguments:

$$\langle \Phi(\hat{x}) \rangle = \langle \Phi(\hat{x}) \rangle_c , \quad \langle \Pi(\hat{x}) \rangle = \langle \Pi(\hat{x}) \rangle_c ,\tag{II.11}$$

$$\begin{aligned}\langle T \Phi(\hat{x}_1) \Phi(\hat{x}_2) \rangle &= \lim_{J, \sigma \rightarrow 0} \frac{\delta}{i \delta J(\hat{x}_1)} \frac{\delta}{i \delta J(\hat{x}_2)} e^{W[J, \sigma]} \\ &= \lim_{J, \sigma \rightarrow 0} \frac{\delta}{i \delta J(\hat{x}_1)} \left\{ \left( \frac{\delta}{i \delta J(\hat{x}_2)} W[J, \sigma] \right) e^{W[J, \sigma]} \right\} \\ &= \lim_{J, \sigma \rightarrow 0} \left\{ \left( \frac{\delta}{i \delta J(\hat{x}_1)} \frac{\delta}{i \delta J(\hat{x}_2)} W[J, \sigma] \right) + \left( \frac{\delta}{i \delta J(\hat{x}_1)} W[J, \sigma] \right) \left( \frac{\delta}{i \delta J(\hat{x}_2)} W[J, \sigma] \right) \right\} e^{W[J, \sigma]} \\ &= \langle T \Phi(\hat{x}_1) \Phi(\hat{x}_2) \rangle_c + \langle \Phi(\hat{x}_1) \rangle \langle \Phi(\hat{x}_2) \rangle ,\end{aligned}\tag{II.12}$$

where  $\langle \cdot \rangle_c$  denotes the connected part of the expectation value. Analogously we obtain

$$\langle T \Pi(\hat{x}_1) \Phi(\hat{x}_2) \rangle = \langle T \Pi(\hat{x}_1) \Phi(\hat{x}_2) \rangle_c + \langle \Pi(\hat{x}_1) \rangle \langle \Phi(\hat{x}_2) \rangle ,$$

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<sup>1</sup>Born Bogoliubov Green Kirkwood Yvon

$$\langle T\Pi(\hat{x}_1)\Pi(\hat{x}_2) \rangle = \langle T\Pi(\hat{x}_1)\Pi(\hat{x}_2) \rangle_c + \langle \Pi(\hat{x}_1) \rangle \langle \Pi(\hat{x}_2) \rangle ,$$

$$\begin{aligned} \langle T\Phi(\hat{x}_1)\Phi(\hat{x}_2)\Phi(\hat{x}_3) \rangle &= \langle T\Phi(\hat{x}_1)\Phi(\hat{x}_2)\Phi(\hat{x}_3) \rangle_c + \langle T\Phi(\hat{x}_1)\Phi(\hat{x}_2) \rangle_c \langle \Phi(\hat{x}_3) \rangle \\ &+ \langle T\Phi(\hat{x}_1)\Phi(\hat{x}_3) \rangle_c \langle \Phi(\hat{x}_2) \rangle + \langle T\Phi(\hat{x}_2)\Phi(\hat{x}_3) \rangle_c \langle \Phi(\hat{x}_1) \rangle + \langle \Phi(\hat{x}_1) \rangle \langle \Phi(\hat{x}_2) \rangle \langle \Phi(\hat{x}_3) \rangle , \quad \dots \end{aligned}$$

(II.13)

The expressions for equal-time Green functions are obtained by taking the well-defined equal-time limit which yields the appropriate operator ordering in the cluster expansions. We arrive at

$$\langle \Phi(x) \rangle = \langle \Phi(x) \rangle_c , \quad \langle \Pi(x) \rangle = \langle \Pi(x) \rangle_c ,$$

$$\langle \Phi(x_1)\Phi(x_2) \rangle = \langle \Phi(x_1)\Phi(x_2) \rangle_c + \langle \Phi(x_1) \rangle \langle \Phi(x_2) \rangle ,$$

$$\langle \Pi(x_1)\Phi(x_2) \rangle = \langle \Pi(x_1)\Phi(x_2) \rangle_c + \langle \Pi(x_1) \rangle \langle \Phi(x_2) \rangle ,$$

$$\langle \Pi(x_1)\Pi(x_2) \rangle = \langle \Pi(x_1)\Pi(x_2) \rangle_c + \langle \Pi(x_1) \rangle \langle \Pi(x_2) \rangle ,$$

$$\begin{aligned} \langle \Phi(x_1)\Phi(x_2)\Phi(x_3) \rangle &= \langle \Phi(x_1)\Phi(x_2)\Phi(x_3) \rangle_c + \langle \Phi(x_1)\Phi(x_2) \rangle_c \langle \Phi(x_3) \rangle \\ &+ \langle \Phi(x_1)\Phi(x_3) \rangle_c \langle \Phi(x_2) \rangle + \langle \Phi(x_2)\Phi(x_3) \rangle_c \langle \Phi(x_1) \rangle + \langle \Phi(x_1) \rangle \langle \Phi(x_2) \rangle \langle \Phi(x_3) \rangle , \quad \dots \end{aligned}$$

(II.14)

where all (equal) time arguments have been suppressed. In view of their length the cluster expansions for the other Green functions required for our calculations are not explicitly given here, but e.g. can be found in [10].

Due to equations (II.4) and (II.5) the 1-point functions in all cluster expansions are now assumed to vanish, which in the end leads to the same result as considering the cluster expansions for Green functions containing the original fields  $\phi$  and  $\pi$  and setting  $\langle \phi \rangle = \Phi_0$  and  $\langle \pi \rangle = 0$ .

The cluster expansions then are truncated by neglecting all connected n-point Green functions with  $n > N$ ; in our case with  $n > 4$ . Inserting the truncated cluster expansions into the equations of motion of type (II.9) up to the equations for the 4-point functions leads to a closed system of coupled nonlinear equations for the connected Green functions (i.e. a system of correlation dynamical equations) up to the 4-point level in analogy to the n-body correlation dynamics in nonrelativistic many-body theory [5]. For this straightforward but somewhat tedious derivation, which in view of its length is not explicitly given here, all truncated cluster expansions up to the 6-point level are required, since the highest order Green functions appearing in the hierarchy equations (II.9) up to the 4-point level are the 6-point functions.

The resulting equations of motion for the connected Green functions still have to be renormalized.  $\Phi^4$ -theory in  $1 + 1$  dimensions is superrenormalizable and only requires a mass renormalization.

There is only one mass counterterm due to the divergent tadpole diagram comprising the contribution to the selfenergy of lowest order in the coupling constant. This mass counterterm can be evaluated analytically either in the framework of perturbation theory or, equivalently, by normal ordering the Hamiltonian with respect to the perturbative vacuum<sup>2</sup>. We obtain [11]:

$$m_0^2 = m^2 + \delta m^2 \quad , \quad \delta m^2 = -3\lambda\Delta(0) \quad (\text{II.15})$$

with

$$\Delta(x_1 - x_2) = \int \frac{dp}{2\pi} \frac{e^{ip(x_1 - x_2)}}{2\omega(p)} \quad , \quad \omega(p) = \sqrt{p^2 + m^2} \quad . \quad (\text{II.16})$$

The logarithmically divergent mass counterterm (II.15) can be analytically removed from the equations of motion by normal ordering all operator products with respect to the perturbative vacuum and thereby splitting off the short-distance singularities of the free equal-time Green functions. The normal ordering within the connected Green functions only takes place within the 2-point functions, since in the other connected n-point functions all field operators commute with each other. For the 2-point functions we have:

$$\begin{aligned} \langle \Phi(x_1)\Phi(x_2) \rangle_c &= \langle : \Phi(x_1)\Phi(x_2) : \rangle_c + \Delta(x_1 - x_2) \quad , \\ \langle \Pi(x_1)\Pi(x_2) \rangle_c &= \langle : \Pi(x_1)\Pi(x_2) : \rangle_c - (\partial_{x_1}^2 - m^2) \Delta(x_1 - x_2) \quad , \\ \langle \Pi(x_1)\Phi(x_2) \rangle_c &= \langle : \Pi(x_1)\Phi(x_2) : \rangle_c - \frac{i}{2}\delta(x_1 - x_2) \quad , \\ \langle \Phi(x_1)\Pi(x_2) \rangle_c &= \langle : \Phi(x_1)\Pi(x_2) : \rangle_c + \frac{i}{2}\delta(x_1 - x_2) \quad . \end{aligned} \quad (\text{II.17})$$

After inserting (II.15) and (II.17) into the equations of motion for the connected Green functions, all terms containing a factor  $\Delta(0)$  mutually cancel out, leading to a closed set of renormalized equations for the normal ordered connected Green functions.

The final step is to transform the renormalized correlation dynamical equations of motion from coordinate space to an arbitrary single particle basis in order to simplify their numerical integration. In this respect we expand the field  $\Phi$  and its conjugate momentum  $\Pi$  according to

$$\Phi(x) = \sum_{\alpha} \Phi_{\alpha} \psi_{\alpha}(x) \quad , \quad \Pi(x) = \sum_{\alpha} \Pi_{\alpha} \psi_{\alpha}(x) \quad \text{with} \quad \int dx \psi_{\alpha}^*(x) \psi_{\beta}(x) = \delta_{\alpha\beta} \quad . \quad (\text{II.18})$$

For the corresponding equal-time Green functions we then have

$$\begin{aligned} \langle \Phi(x_1)\Phi(x_2) \rangle_c &= \sum_{\alpha\beta} \langle \Phi_{\alpha}\Phi_{\beta} \rangle_c \psi_{\alpha}(x_1) \psi_{\beta}(x_2) = \langle : \Phi(x_1)\Phi(x_2) : \rangle_c + \Delta(x_1 - x_2) \\ &= \sum_{\alpha\beta} \langle : \Phi_{\alpha}\Phi_{\beta} : \rangle_c \psi_{\alpha}(x_1) \psi_{\beta}(x_2) + \sum_{\alpha\beta} \Delta_{\alpha\beta} \psi_{\alpha}(x_1) \psi_{\beta}(x_2) \quad , \end{aligned}$$

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<sup>2</sup>In addition to introducing the mass counterterm, one of course has to subtract the constant infinite zero-point contribution from the Hamiltonian, which however does not enter the Heisenberg equations.

$$\langle \Phi(x_1)\Phi(x_2)\Phi(x_3) \rangle = \sum_{\alpha\beta\gamma} \langle \Phi_\alpha\Phi_\beta\Phi_\gamma \rangle \psi_\alpha(x_1)\psi_\beta(x_2)\psi_\gamma(x_3) ,$$

$$\langle \Phi(x_1)\Phi(x_2)\Phi(x_3) \rangle_c = \sum_{\alpha\beta\gamma} \langle \Phi_\alpha\Phi_\beta\Phi_\gamma \rangle_c \psi_\alpha(x_1)\psi_\beta(x_2)\psi_\gamma(x_3) , \quad \dots . \quad (\text{II.19})$$

The corresponding expressions for the other Green functions are obtained analogously. By inserting the expansions (II.19) into the renormalized equations of motion and projecting out the matrix elements with respect to the single particle basis we obtain the final result for the correlation dynamical equations of motion for the  $\Phi_{1+1}^4$ -theory, which are denoted by  $\Phi^4CD$  ( $\Phi^4$  Correlation Dynamics) furtheron, and can directly be used for a numerical integration. In view of their length the  $\Phi^4CD$  equations are shifted to appendix A.

### III. APPLICATION TO GROUND STATE SYMMETRY BREAKING

In this section we apply the  $\Phi^4CD$  equations to the determination of the effective potential of the  $\Phi_{1+1}^4$ -theory at zero temperature<sup>3</sup> and thereby investigate the spontaneous breakdown of the symmetry under the discrete transformation  $\phi \rightarrow -\phi$  for values of the coupling exceeding a critical value, which manifests itself in a nonzero ground state magnetization  $\langle \phi \rangle$ .

Since we also want to study the influence of the different connected n-point functions, we introduce 4 limiting cases of the correlation dynamical equations; these are denoted as  $\Phi^4CD(2)$ ,  $\Phi^4CD(2,3)$ ,  $\Phi^4CD(2,4)$  and  $\Phi^4CD(2,3,4)$ , where the numbers in parantheses are the orders of connected n-point functions that are taken into account (i.e., for instance in the  $\Phi^4CD(2,4)$  case the connected 3-point functions are set equal to zero and  $\Phi^4CD(2,3,4)$  denotes the original  $\Phi^4CD$  approximation discussed in section II).

In order to integrate equations (A.1) - (A.12) numerically, we choose plane waves in a one-dimensional box with periodic boundary conditions as a single particle basis, i.e. we work in discretized momentum space. In order to select an appropriate box size for a given renormalized mass, we compare the numerically obtained GEP<sup>4</sup> solution for the ground state configuration within the discretized system to the analytically accessible GEP solution in the continuum limit (for the GEP approximation, see appendix B or [11–13]). Due to the large amount of computer time needed for  $\Phi^4CD$  calculations we have to compromise between a good momentum space resolution and the convergence with a minimum number of plane waves. It has proven to be most effective to choose a box size of 100 fm for a renormalized mass of 10 MeV. In fig. 1 we show the ground state magnetization  $\langle \phi \rangle = \Phi_0$  as a function of the dimensionless coupling  $\lambda/4m^2$  for these parameters, where different numbers of single particle states have been taken into account. The vertical line shows the position of the critical coupling for symmetry breaking in the continuum

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<sup>3</sup>i.e. the ground state energy density in a subspace with a given magnetization

<sup>4</sup>**G**aussian **E**ffective **P**otential

limit. For practical purposes we will always use the 15 lowest lying plane waves in the following calculations, unless explicitly stated otherwise. Within the GEP approximation we then obtain a critical coupling of  $\lambda_{crit}/4m^2 = 2.568$  for the discretized system as compared to a critical coupling of  $\lambda_{crit}/4m^2 = 2.5527045$  in the continuum limit.

Since the  $\Phi^4CD$  equations only describe the propagation of equal-time Green functions for a given initial configuration, they cannot directly be applied to the evaluation of static equilibrium properties of the system. As in the case of correlation dynamics for nonrelativistic many-body theory there is no easy access to the stationary solutions of the equations, and moreover it is not clear, in how far additional constraints have to be imposed on the subspace of stationary configurations in order to select only the physical solutions.

We therefore use a different approach for the evaluation of equal-time ground state Green functions within the  $\Phi^4CD$  approximation. Starting with the trivial exact ground state configuration for  $\lambda = 0$  and a given fixed value of  $\Phi_0$  as an initial condition, we continuously switch on the coupling while propagating the system in time. The time-dependence of the coupling is chosen to be linear with  $\lambda = \beta t$  and  $\beta = const.$

In fig. 2 the energy density obtained within this time-dependent method is shown as a function of the coupling for different values of  $\beta$  for  $\Phi_0 = 0.4$  and all 4 limiting cases of the correlation dynamical equations. In all cases an asymptotic curve is approached with decreasing  $\beta$ , i.e. when the coupling is switched on more slowly. This indicates that in the limit  $\beta \rightarrow 0$  the whole process becomes fully adiabatic, i.e. the system will time-dependently follow the trajectory of the ground state as a function of the coupling. For the  $\Phi^4CD(2)$  approximation, taking into account only the connected 2-point functions, which is the field theoretical analogon to time-dependent Hartree-Fock theory, we have direct access to the static ground state solutions since its stationary limit is simply given by the GEP approximation. The corresponding curves are shown in the upper left part of fig. 2; the GEP is displayed as the lower one of the two solid lines. Indeed, the asymptotic curve of the time-dependent method is identical to the GEP solution. In addition, we have checked numerically that for all 4 limiting cases of correlation dynamics the system increasingly equilibrates along its trajectory with decreasing  $\beta$ , i.e. if we stop increasing  $\lambda$  at some point in time, the system will remain in its present state when propagated further.

In choosing a finite value for  $\beta$  we have to compromise between a good convergence of our time-dependent method and a minimum of computer time we want to invest; the curves in the two following pictures have therefore been evaluated with  $\beta = 5\text{MeV}^2c/\text{fm}$ .

The ground state energy density for various values of  $\Phi_0$  is plotted versus the coupling  $\lambda/4m^2$  in fig. 3 for all 4 limiting cases of correlation dynamics, where for the reasons mentioned above the  $\Phi^4CD(2)$  approximation has been replaced by the GEP approximation. The GEP and the  $\Phi^4CD(2,4)$  approximation each predict a first order phase transition, since the first curve to intersect the  $\Phi_0 = 0$  (i.e. the symmetric phase) energy density in the GEP case has  $\Phi_0 = 0.6$  and in the  $\Phi^4CD(2,4)$  case has  $\Phi_0 = 0.8$ ; i.e. they both lead to a finite value of  $\Phi_0$ , implying that the vacuum magnetization has to jump to that value discontinuously at the critical value of the

coupling constant (compare fig. 1 for the GEP). In contrast to that, the  $\Phi^4 CD(2, 3)$  approximation and the  $\Phi^4 CD(2, 3, 4)$  approximation each predict a second order phase transition, since in both cases all curves with  $\Phi_0 \neq 0$  intersect the curve with  $\Phi_0 = 0$  in the correct order, thus enabling the vacuum magnetization to increase continuously once the critical coupling is exceeded.

At this point it is useful to recall, that there is a rigorous mathematical proof of the statement that there can be no first order phase transition in the  $\Phi_{1+1}^4$ -theory (in the continuum limit) [14]. This proof is based on the Simon-Griffith theorem [15], which in turn is obtained by considering the  $\Phi^4$  field theory as a proper limit of a generalized Ising model.

Thus we conclude, that in order to describe the order of the phase transition correctly, the inclusion of the connected 3-point function is required. A simple "geometrical" explanation of this fact will be given in the discussion below.

In fig. 4 the effective potential is plotted as a function of the magnetization  $\Phi_0$  for various values of the coupling. The same data have been used as for the previous figure. In this representation, the first order nature of the phase transition in the GEP and the  $\Phi^4 CD(2, 4)$  case can be seen from the fact, that for these approximations there is a maximum in the effective potential between the minimum at  $\Phi_0 = 0$  and the second minimum on the right hand side (for the GEP this minimum is not very pronounced); thus at the critical coupling the second minimum has to be located at a finite value of  $\Phi_0$ . In the case of the  $\Phi^4 CD(2, 3)$  and the  $\Phi^4 CD(2, 3, 4)$  approximation there is no such intermediate maximum, and as the critical coupling is approached from above, the second minimum is shifted to the left towards  $\Phi_0 = 0$ .

For  $\Phi_0 \rightarrow \infty$  the system will approach the classical limit, i.e. with decreasing  $\Phi_0$  higher order correlations increasingly become important; the inclusion of correlations always lowers the energy density, since a larger configuration space is opened up for the system. However, for obvious mathematical reasons the inclusion of the 2-point function alone cannot lower the energy density at  $\Phi_0 = 0$ ; unlike the other n-point functions of even order the 2-point function therefore is most important in the region on the right hand side in the plots of fig. 4. The 4-point function, as the highest order correlation included in this work, is most important around  $\Phi_0 = 0$ , i.e. for small classical field strengths. The intermediate maximum now appears in between the domains governed by the 2-point function on the right and the 4-point function on the left, an inclusion of higher correlations of even order would most probably lower the energy density in a region even more concentrated around  $\Phi_0 = 0$  and therefore not cure this problem. The inclusion of connected n-point functions of uneven order cures the problem, since for symmetry reasons these have to vanish as  $\Phi_0 \rightarrow 0$ . The 3-point function therefore obviously assumes its main importance exactly in the region in between the domains of the 2- and the 4-point function, it "bends down" the intermediate maximum.

Finally, in table I we give the values for the critical coupling extracted from our calculations for different numerical parameters and the 4 different limiting cases of  $\Phi^4 CD$ . The critical couplings  $\lambda_{crit}/4m^2$  obtained by other authors, that are also using non-perturbative techniques adopted from standard many-body theory, are as follows: [16,17] obtain 1.829 or 1.375, respectively, using the



discretized light-front quantization method, [18] obtain 1.72 using a varitional approximation with trial states which are quartic rather than gaussians (quadratic exponentials) causing an inclusion of the 3-field correlation amplitude in the language of the coupled cluster method [1], [2] obtain the estimate  $0.95 < \lambda/4m^2 < 2.15$  using an improved version of the coupled cluster expansion method, and using second order perturbation theory in the residual interaction leads to 1.14 [19]. All of these authors obtain a second order phase transition in agreement with the Simon-Griffith-theorem.

approximation	No. of states	$\beta$ [MeV <sup>2</sup> c/fm]	$\lambda_{crit}/4m^2$	order of phasetransition
GEP	15	-	2.568	first
$\Phi^4 CD(2,3)$	11	5.0	1.679	second
	13	5.0	1.649	
	15	5.0	1.629	
	17	5.0	1.613	
	19	5.0	1.601	
$\Phi^4 CD(2,4)$	15	5.0	3.81	first
$\Phi^4 CD(2,3,4)$	15	5.0	2.446	second

TABLE I. Values for the critical coupling for different approximations and numerical parameters.

#### IV. SUMMARY AND OUTLOOK

This work comprises the first application of correlation dynamics for equal-time Green functions to a field theoretical problem, i.e. to the determination of the effective potential in  $\Phi^4_{1+1}$ -theory. After giving a derivation of the corresponding equations of motion, we showed that we are able to evaluate equal-time quantities in the interacting ground state for a given vacuum magnetization by time-dependently increasing the coupling in an adiabatic process.

Our numerical results predict a second order phase transition in agreement with the Simon-Griffith theorem, as soon as the connected 3-point function is included. However, since the connected 2-point function and the connected 3-point function alone are not able to lower the energy of the symmetric phase, the critical coupling obtained within the  $\Phi^4 CD(2, 3)$  approximation is too low. Going one step further in our expansion, we find that the connected 4-point function in the  $\Phi^4 CD(2, 3, 4)$  method increases the coupling to a value, which is only slightly lower than the value obtained by the GEP approximation, where however the shape of the effective potential changes completely as compared to the GEP.

In general, the results of this work demonstrate the applicability of correlation dynamics to the description of low-energy (ground state) phenomena in local field theories. Since in principle the equations are designed to describe the propagation of the system in time with arbitrary initial conditions, our method is also a potentially powerful tool for the investigation of non-equilibrium properties of relativistic field theories, e.g. the response to external perturbations.

The present study can be viewed as a first step towards the application of correlation dynamics to  $SU(N)$  gauge theories, aiming at a non-perturbative description of the infrared behaviour of QCD; we already presented the corresponding equations of motion in [10].

#### APPENDIX A: EQUATIONS OF MOTION OF $\Phi^4$ CORRELATION DYNAMICS

In this appendix we present the renormalized correlation dynamical equations for the normal ordered connected Green functions up to the 4-point level, formulated with respect to an arbitrary single particle basis set (see section II). In order to compactify the equations we introduce the following abbreviations:

$$\langle \alpha | \lambda_1 \lambda_2 \rangle = \int dx \, \psi_\alpha^*(x) \psi_{\lambda_1}(x) \psi_{\lambda_2}(x) ,$$

$$\langle \alpha | \lambda_1 \lambda_2 \lambda_3 \rangle = \int dx \, \psi_\alpha^*(x) \psi_{\lambda_1}(x) \psi_{\lambda_2}(x) \psi_{\lambda_3}(x) ,$$

$$\langle \alpha \beta \gamma | 1 \rangle = \int dx \, \psi_\alpha^*(x) \psi_\beta^*(x) \psi_\gamma^*(x) ,$$

$$\langle \alpha \beta \gamma | \lambda \rangle = \int dx \, \psi_\alpha^*(x) \psi_\beta^*(x) \psi_\gamma^*(x) \psi_\lambda(x) ,$$

$$t_{\alpha\beta} = \int dx \psi_{\alpha}^*(x) (\partial_x^2 - m^2 - 3\lambda\Phi_0^2) \psi_{\beta}(x) ,$$

$$U_{\alpha\beta} = -3\lambda \sum_{\lambda_1\lambda_2} \langle \alpha | \lambda_1 \lambda_2 \beta \rangle \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} : \rangle_c .$$

The permutation operator interchanging the indices  $\alpha$  and  $\beta$  is denoted by  $\mathcal{P}_{\alpha\beta}$ . Although normal ordering only affects the connected 2-point functions, we write out the normal ordering operation  $: \cdot :$  in all connected Green functions in order to have a uniform notation. The equations then read:

$$\frac{d}{dt} \langle : \Phi_{\alpha} \Phi_{\beta} : \rangle_c = \langle : \Pi_{\alpha} \Phi_{\beta} : \rangle_c + \langle : \Phi_{\alpha} \Pi_{\beta} : \rangle_c , \quad (\text{A.1})$$

$$\begin{aligned} \frac{d}{dt} \langle : \Pi_{\alpha} \Phi_{\beta} : \rangle_c &= \langle : \Pi_{\alpha} \Pi_{\beta} : \rangle_c \\ &+ \sum_{\lambda} (t_{\alpha\lambda} + U_{\alpha\lambda}) \langle : \Phi_{\lambda} \Phi_{\beta} : \rangle_c + \sum_{\lambda} (U_{\alpha\lambda} - 3\lambda\Phi_0^2 \delta_{\alpha\lambda}) \Delta_{\lambda\beta} \\ &- \lambda \sum_{\lambda_1\lambda_2\lambda_3} \langle \alpha | \lambda_1 \lambda_2 \lambda_3 \rangle \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Phi_{\lambda_3} \Phi_{\beta} : \rangle_c \\ &- 3\lambda\Phi_0 \sum_{\lambda_1\lambda_2} \langle \alpha | \lambda_1 \lambda_2 \rangle \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Phi_{\beta} : \rangle_c , \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} \frac{d}{dt} \langle : \Pi_{\alpha} \Pi_{\beta} : \rangle_c &= (1 + \mathcal{P}_{\alpha\beta}) \sum_{\lambda} (t_{\alpha\lambda} + U_{\alpha\lambda}) \langle : \Phi_{\lambda} \Pi_{\beta} : \rangle_c \\ &- \lambda(1 + \mathcal{P}_{\alpha\beta}) \sum_{\lambda_1\lambda_2\lambda_3} \langle \alpha | \lambda_1 \lambda_2 \lambda_3 \rangle \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Phi_{\lambda_3} \Pi_{\beta} : \rangle_c \\ &- 3\lambda\Phi_0(1 + \mathcal{P}_{\alpha\beta}) \sum_{\lambda_1\lambda_2} \langle \alpha | \lambda_1 \lambda_2 \rangle \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Pi_{\beta} : \rangle_c , \end{aligned} \quad (\text{A.3})$$

$$\frac{d}{dt} \langle : \Phi_{\alpha} \Phi_{\beta} \Phi_{\gamma} : \rangle_c = \langle : \Pi_{\alpha} \Phi_{\beta} \Phi_{\gamma} : \rangle_c + \langle : \Phi_{\alpha} \Pi_{\beta} \Phi_{\gamma} : \rangle_c + \langle : \Phi_{\alpha} \Phi_{\beta} \Pi_{\gamma} : \rangle_c , \quad (\text{A.4})$$

$$\begin{aligned} \frac{d}{dt} \langle : \Pi_{\alpha} \Phi_{\beta} \Phi_{\gamma} : \rangle_c &= \langle : \Pi_{\alpha} \Pi_{\beta} \Phi_{\gamma} : \rangle_c + \langle : \Pi_{\alpha} \Phi_{\beta} \Pi_{\gamma} : \rangle_c \\ &+ \sum_{\lambda} (t_{\alpha\lambda} + U_{\alpha\lambda}) \langle : \Phi_{\lambda} \Phi_{\beta} \Phi_{\gamma} : \rangle_c \\ &- 3\lambda \sum_{\lambda_1\lambda_2\lambda_3} \langle \alpha | \lambda_1 \lambda_2 \lambda_3 \rangle (1 + \mathcal{P}_{\beta\gamma}) \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Phi_{\beta} : \rangle_c (\langle : \Phi_{\lambda_3} \Phi_{\gamma} : \rangle_c + \Delta_{\lambda_3\gamma}) \\ &- 3\lambda\Phi_0 \sum_{\lambda_1\lambda_2} \langle \alpha | \lambda_1 \lambda_2 \rangle \{ \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Phi_{\beta} \Phi_{\gamma} : \rangle_c \\ &+ 2(\langle : \Phi_{\lambda_1} \Phi_{\beta} : \rangle_c + \Delta_{\lambda_1\beta})(\langle : \Phi_{\lambda_2} \Phi_{\gamma} : \rangle_c + \Delta_{\lambda_2\gamma}) \} , \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned}
\frac{d}{dt}\langle : \Pi_\alpha \Pi_\beta \Phi_\gamma : \rangle_c &= \langle : \Pi_\alpha \Pi_\beta \Pi_\gamma : \rangle_c \\
&+ (1 + \mathcal{P}_{\alpha\beta}) \sum_\lambda (t_{\alpha\lambda} + U_{\alpha\lambda}) \langle : \Phi_\lambda \Pi_\beta \Phi_\gamma : \rangle_c \\
&- 3\lambda(1 + \mathcal{P}_{\alpha\beta}) \sum_{\lambda_1 \lambda_2 \lambda_3} \langle \alpha | \lambda_1 \lambda_2 \lambda_3 \rangle \{ \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Pi_\beta : \rangle_c (\langle : \Phi_{\lambda_3} \Phi_\gamma : \rangle_c + \Delta_{\lambda_3 \gamma}) \\
&+ \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Phi_\gamma : \rangle_c \langle : \Phi_{\lambda_3} \Pi_\beta : \rangle_c \} \\
&- 3\lambda \Phi_0 (1 + \mathcal{P}_{\alpha\beta}) \sum_{\lambda_1 \lambda_2} \langle \alpha | \lambda_1 \lambda_2 \rangle \{ \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Pi_\beta \Phi_\gamma : \rangle_c \\
&+ 2 \langle : \Phi_{\lambda_1} \Pi_\beta : \rangle_c (\langle : \Phi_{\lambda_2} \Phi_\gamma : \rangle_c + \Delta_{\lambda_2 \gamma}) \} , \tag{A.6}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\langle : \Pi_\alpha \Pi_\beta \Pi_\gamma : \rangle_c &= (1 + \mathcal{P}_{\alpha\beta} + \mathcal{P}_{\alpha\gamma}) \sum_\lambda (t_{\alpha\lambda} + U_{\alpha\lambda}) \langle : \Phi_\lambda \Pi_\beta \Pi_\gamma : \rangle_c \\
&- 3\lambda(1 + \mathcal{P}_{\alpha\beta} + \mathcal{P}_{\alpha\gamma}) \sum_{\lambda_1 \lambda_2 \lambda_3} \langle \alpha | \lambda_1 \lambda_2 \lambda_3 \rangle (1 + \mathcal{P}_{\beta\gamma}) \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Pi_\beta : \rangle_c \langle : \Phi_{\lambda_3} \Pi_\gamma : \rangle_c \\
&- 3\lambda \Phi_0 (1 + \mathcal{P}_{\alpha\beta} + \mathcal{P}_{\alpha\gamma}) \sum_{\lambda_1 \lambda_2} \langle \alpha | \lambda_1 \lambda_2 \rangle \{ \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Pi_\beta \Pi_\gamma : \rangle_c + 2 \langle : \Phi_{\lambda_1} \Pi_\beta : \rangle_c \langle : \Phi_{\lambda_2} \Pi_\gamma : \rangle_c \} \\
&+ \frac{3}{2} \lambda \Phi_0 \langle \alpha \beta \gamma | 1 \rangle , \tag{A.7}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\langle : \Phi_\alpha \Phi_\beta \Phi_\gamma \Phi_\delta : \rangle_c &= \langle : \Pi_\alpha \Phi_\beta \Phi_\gamma \Phi_\delta : \rangle_c \\
&+ \langle : \Phi_\alpha \Pi_\beta \Phi_\gamma \Phi_\delta : \rangle_c + \langle : \Phi_\alpha \Phi_\beta \Pi_\gamma \Phi_\delta : \rangle_c + \langle : \Phi_\alpha \Phi_\beta \Phi_\gamma \Pi_\delta : \rangle_c , \tag{A.8}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\langle : \Pi_\alpha \Phi_\beta \Phi_\gamma \Phi_\delta : \rangle_c &= \langle : \Pi_\alpha \Pi_\beta \Phi_\gamma \Phi_\delta : \rangle_c + \langle : \Pi_\alpha \Phi_\beta \Pi_\gamma \Phi_\delta : \rangle_c + \langle : \Pi_\alpha \Phi_\beta \Phi_\gamma \Pi_\delta : \rangle_c \\
&+ \sum_\lambda (t_{\alpha\lambda} + U_{\alpha\lambda}) \langle : \Phi_\lambda \Phi_\beta \Phi_\gamma \Phi_\delta : \rangle_c \\
&- 3\lambda \sum_{\lambda_1 \lambda_2 \lambda_3} \langle \alpha | \lambda_1 \lambda_2 \lambda_3 \rangle \{ 2(\langle : \Phi_{\lambda_1} \Phi_\beta : \rangle_c + \Delta_{\lambda_1 \beta})(\langle : \Phi_{\lambda_2} \Phi_\gamma : \rangle_c + \Delta_{\lambda_2 \gamma})(\langle : \Phi_{\lambda_3} \Phi_\delta : \rangle_c + \Delta_{\lambda_3 \delta}) \\
&+ (1 + \mathcal{P}_{\beta\gamma} + \mathcal{P}_{\beta\delta})[(\langle : \Phi_{\lambda_1} \Phi_\beta : \rangle_c + \Delta_{\lambda_1 \beta}) \langle : \Phi_{\lambda_2} \Phi_{\lambda_3} \Phi_\gamma \Phi_\delta : \rangle_c \\
&+ \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Phi_\beta : \rangle_c \langle : \Phi_{\lambda_3} \Phi_\gamma \Phi_\delta : \rangle_c] \} \\
&- 6\lambda \Phi_0 \sum_{\lambda_1 \lambda_2} \langle \alpha | \lambda_1 \lambda_2 \rangle (1 + \mathcal{P}_{\beta\gamma} + \mathcal{P}_{\beta\delta})(\langle : \Phi_{\lambda_1} \Phi_\beta : \rangle_c + \Delta_{\lambda_1 \beta}) \langle : \Phi_{\lambda_2} \Phi_\gamma \Phi_\delta : \rangle_c , \tag{A.9}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \langle : \Pi_\alpha \Pi_\beta \Phi_\gamma \Phi_\delta : \rangle_c &= \langle : \Pi_\alpha \Pi_\beta \Pi_\gamma \Phi_\delta : \rangle_c + \langle : \Pi_\alpha \Pi_\beta \Phi_\gamma \Pi_\delta : \rangle_c \\
&+ (1 + \mathcal{P}_{\alpha\beta}) \sum_\lambda (t_{\alpha\lambda} + U_{\alpha\lambda}) \langle : \Phi_\lambda \Pi_\beta \Phi_\gamma \Phi_\delta : \rangle_c \\
&- 3\lambda(1 + \mathcal{P}_{\alpha\beta}) \sum_{\lambda_1 \lambda_2 \lambda_3} \langle \alpha | \lambda_1 \lambda_2 \lambda_3 \rangle \{ 2 \langle : \Phi_{\lambda_1} \Pi_\beta : \rangle_c (\langle : \Phi_{\lambda_2} \Phi_\gamma : \rangle_c + \Delta_{\lambda_2 \gamma}) (\langle : \Phi_{\lambda_3} \Phi_\delta : \rangle_c + \Delta_{\lambda_3 \delta}) \\
&+ \langle : \Phi_{\lambda_1} \Pi_\beta : \rangle_c \langle : \Phi_{\lambda_2} \Phi_{\lambda_3} \Phi_\gamma \Phi_\delta : \rangle_c + (1 + \mathcal{P}_{\gamma\delta}) (\langle : \Phi_{\lambda_1} \Phi_\gamma : \rangle_c + \Delta_{\lambda_1 \gamma}) \langle : \Phi_{\lambda_2} \Phi_{\lambda_3} \Pi_\beta \Phi_\delta : \rangle_c \\
&+ (1 + \mathcal{P}_{\beta\gamma} + \mathcal{P}_{\beta\delta}) \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Pi_\beta : \rangle_c \langle : \Phi_{\lambda_3} \Phi_\gamma \Phi_\delta : \rangle_c \} \\
&- 6\lambda\Phi_0(1 + \mathcal{P}_{\alpha\beta}) \sum_{\lambda_1 \lambda_2} \langle \alpha | \lambda_1 \lambda_2 \rangle \{ \langle : \Phi_{\lambda_1} \Pi_\beta : \rangle_c \langle : \Phi_{\lambda_2} \Phi_\gamma \Phi_\delta : \rangle_c \\
&+ (1 + \mathcal{P}_{\gamma\delta}) (\langle : \Phi_{\lambda_1} \Phi_\gamma : \rangle_c + \Delta_{\lambda_1 \gamma}) \langle : \Phi_{\lambda_2} \Pi_\beta \Phi_\delta : \rangle_c \} , \tag{A.10}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \langle : \Pi_\alpha \Pi_\beta \Pi_\gamma \Phi_\delta : \rangle_c &= \langle : \Pi_\alpha \Pi_\beta \Pi_\gamma \Pi_\delta : \rangle_c \\
&+ (1 + \mathcal{P}_{\alpha\beta} + \mathcal{P}_{\alpha\gamma}) \sum_\lambda (t_{\alpha\lambda} + U_{\alpha\lambda}) \langle : \Phi_\lambda \Pi_\beta \Pi_\gamma \Phi_\delta : \rangle_c \\
&- 3\lambda(1 + \mathcal{P}_{\alpha\beta} + \mathcal{P}_{\alpha\gamma}) \sum_{\lambda_1 \lambda_2 \lambda_3} \langle \alpha | \lambda_1 \lambda_2 \lambda_3 \rangle \{ 2 \langle : \Phi_{\lambda_1} \Pi_\beta : \rangle_c \langle : \Phi_{\lambda_2} \Pi_\gamma : \rangle_c (\langle : \Phi_{\lambda_3} \Phi_\delta : \rangle_c + \Delta_{\lambda_3 \delta}) \\
&+ (1 + \mathcal{P}_{\beta\gamma}) \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Pi_\beta \Phi_\delta : \rangle_c \langle : \Phi_{\lambda_3} \Pi_\gamma : \rangle_c + \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Pi_\beta \Pi_\gamma : \rangle_c (\langle : \Phi_{\lambda_3} \Phi_\delta : \rangle_c + \Delta_{\lambda_3 \delta}) \\
&+ (1 + \mathcal{P}_{\beta\gamma} + \mathcal{P}_{\beta\delta}) \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Pi_\beta : \rangle_c \langle : \Phi_{\lambda_3} \Pi_\gamma \Phi_\delta : \rangle_c \} \\
&- 6\lambda\Phi_0(1 + \mathcal{P}_{\alpha\beta} + \mathcal{P}_{\alpha\gamma}) \sum_{\lambda_1 \lambda_2} \langle \alpha | \lambda_1 \lambda_2 \rangle \{ (1 + \mathcal{P}_{\beta\gamma}) \langle : \Phi_{\lambda_1} \Pi_\beta : \rangle_c \langle : \Phi_{\lambda_2} \Pi_\gamma \Phi_\delta : \rangle_c \\
&+ \langle : \Phi_{\lambda_1} \Pi_\beta \Pi_\gamma : \rangle_c (\langle : \Phi_{\lambda_2} \Phi_\delta : \rangle_c + \Delta_{\lambda_2 \delta}) \} \\
&+ \frac{3}{2} \lambda \sum_\lambda \langle \alpha \beta \gamma | \lambda \rangle (\langle : \Phi_\lambda \Phi_\delta : \rangle_c + \Delta_{\lambda \delta}) , \tag{A.11}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \langle : \Pi_\alpha \Pi_\beta \Pi_\gamma \Pi_\delta : \rangle_c &= (1 + \mathcal{P}_{\alpha\beta} + \mathcal{P}_{\alpha\gamma} + \mathcal{P}_{\alpha\delta}) \sum_\lambda (t_{\alpha\lambda} + U_{\alpha\lambda}) \langle : \Phi_\lambda \Pi_\beta \Pi_\gamma \Pi_\delta : \rangle_c \\
&- 3\lambda(1 + \mathcal{P}_{\alpha\beta} + \mathcal{P}_{\alpha\gamma} + \mathcal{P}_{\alpha\delta}) \sum_{\lambda_1 \lambda_2 \lambda_3} \langle \alpha | \lambda_1 \lambda_2 \lambda_3 \rangle \{ 2 \langle : \Phi_{\lambda_1} \Pi_\beta : \rangle_c \langle : \Phi_{\lambda_2} \Pi_\gamma : \rangle_c \langle : \Phi_{\lambda_3} \Pi_\delta : \rangle_c \\
&+ (1 + \mathcal{P}_{\beta\gamma} + \mathcal{P}_{\beta\delta}) [\langle : \Phi_{\lambda_1} \Pi_\beta : \rangle_c \langle : \Phi_{\lambda_2} \Phi_{\lambda_3} \Pi_\gamma \Pi_\delta : \rangle_c + \langle : \Phi_{\lambda_1} \Phi_{\lambda_2} \Pi_\beta : \rangle_c \langle : \Phi_{\lambda_3} \Pi_\gamma \Pi_\delta : \rangle_c] \} \\
&- 6\lambda\Phi_0(1 + \mathcal{P}_{\alpha\beta} + \mathcal{P}_{\alpha\gamma} + \mathcal{P}_{\alpha\delta}) \sum_{\lambda_1 \lambda_2} \langle \alpha | \lambda_1 \lambda_2 \rangle (1 + \mathcal{P}_{\beta\gamma} + \mathcal{P}_{\beta\delta}) \langle : \Phi_{\lambda_1} \Pi_\beta : \rangle_c \langle : \Phi_{\lambda_2} \Pi_\gamma \Pi_\delta : \rangle_c \\
&+ \frac{3}{2} \lambda (1 + \mathcal{P}_{\alpha\beta} + \mathcal{P}_{\alpha\gamma} + \mathcal{P}_{\alpha\delta}) \sum_\lambda \langle \beta \gamma \delta | \lambda \rangle \langle : \Phi_\lambda \Pi_\alpha : \rangle_c . \tag{A.12}
\end{aligned}$$

## APPENDIX B: THE GAUSSIAN EFFECTIVE POTENTIAL FOR $\Phi_{1+1}^4$ -THEORY

The Hamiltonian for the  $\Phi_{1+1}^4$ -system is given by

$$H = \frac{1}{2} \int dx \left[ \pi^2 + \left( \frac{d\phi}{dx} \right)^2 + m_0^2 \phi^2 + \frac{\lambda}{4} \phi^4 \right] \quad (\text{B.1})$$

with  $m_0^2 = m^2 + \delta m^2$ ,  $\delta m^2 = -3\lambda\Delta(0)$  (see section II). Let furthermore  $\phi = \Phi_0 + \Phi$ ,

$$\Phi(x) = (2\pi)^{-\frac{1}{2}} \int dp (2\omega_p)^{-\frac{1}{2}} \left[ a_p + a_{-p}^\dagger \right] e^{ipx} ,$$

$$\omega_p = \sqrt{p^2 + m^2} , \quad a_p|0\rangle = 0 ,$$

where  $|0\rangle$  is the perturbative vacuum. As in section II,  $\Phi_0$  denotes the constant vacuum magnetization, i.e. the ground state expectation value of  $\phi$ .

The GEP approximation consists in the ansatz

$$H_0 = \frac{1}{2} \int dx \left[ \pi^2 + \left( \frac{d\phi}{dx} \right)^2 + M^2 (\phi - \Phi_0)^2 \right] \quad (\text{B.2})$$

for the variational Hamiltonian, i.e. the interacting system is approximated by a free system with an effective mass  $M$ , which serves as a variational parameter. One then has to minimize the expectation value of  $H$  with respect to the ground state of  $H_0$ .

This method is equivalent to a BCS calculation, where the variational wavefunction is given by a boson pair condensate

$$|\Psi\rangle = \mathcal{N} \exp(-S)|0\rangle , \quad S = \frac{1}{2} \int dp \mu(p) a_p^\dagger a_{-p}^\dagger ; \quad (\text{B.3})$$

here  $\mathcal{N}$  denotes a normalization constant and  $\mu(p)$  is the variational parameter.  $|\Psi\rangle$  is a quasiparticle vacuum, i.e.  $b_p|\Psi\rangle = 0$ , where the  $b_p$  can be obtained from the  $a_p$  by means of a Bogoliubov-transformation. We have

$$\Phi(x) = (2\pi)^{-\frac{1}{2}} \int dp (2\Omega_p)^{-\frac{1}{2}} \left[ b_p + b_{-p}^\dagger \right] e^{ipx} \quad (\text{B.4})$$

with  $\Omega_p = \omega_p \frac{1+\mu(p)}{1-\mu(p)}$ .

After minimizing the energy functional, the GEP ansatz yields the equation

$$M^2 - m^2 = 3\lambda \left\{ \Phi_0^2 + \int_{-\infty}^{\infty} \frac{dp}{2\pi} \left( \frac{1}{2\Omega_p} - \frac{1}{2\omega_p} \right) \right\} \quad (\text{B.5})$$

with  $\Omega_p = \sqrt{p^2 + M^2}$  for the effective mass  $M$ . Equation (B.5) is identical to the HFDS<sup>5</sup> equation and the Gap equation following from the BCS ansatz [11].

For known effective mass  $M$ , the equal-time Green functions can be evaluated via (B.4) as expectation values with respect to the quasiparticle vacuum  $|\Psi\rangle$ . The value for the energy density functional as a function of  $\Phi_0$  (obtained with the method described above) is the Gaussian approximation for the effective potential of the theory.

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## Figure Captions

fig.1: Vacuum magnetization as a function of the coupling in the GEP approximation for different numbers of basis states.

fig.2: Ground state energy density during the time-dependent process as a function of the coupling for a given vacuum magnetization of  $\Phi_0 = 0.4$  and different values of  $\beta$ ; upper left:  $\Phi^4CD(2)$ , upper right:  $\Phi^4CD(2,3)$ , lower left:  $\Phi^4CD(2,4)$ , lower right:  $\Phi^4CD(2,3,4)$ ; in addition the static GEP curve is displayed in the upper left plot.

fig.3: Ground state energy density as a function of the coupling for different vacuum magnetizations; all curves except the one in GEP approximation have been evaluated time-dependently with  $\beta = 5 \text{ MeV}^2c/\text{fm}$ ; upper left: GEP, upper right:  $\Phi^4CD(2,3)$ , lower left:  $\Phi^4CD(2,4)$ , lower right:  $\Phi^4CD(2,3,4)$ .

fig.4: Effective potential as a function of the vacuum magnetization for different values of the coupling; all curves except the one in GEP approximation have been evaluated time-dependently with  $\beta = 5 \text{ MeV}^2c/\text{fm}$ ; upper left: GEP, upper right:  $\Phi^4CD(2,3)$ , lower left:  $\Phi^4CD(2,4)$ , lower right:  $\Phi^4CD(2,3,4)$ .



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